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Neutrino Radiation Fields in General Relativity

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Abstract

All space-times admitting a neutrino radiation field are obtained. Three classes of such space-times exist, characterised by the Weyl tensor being of type D, N (or O) or III.

1. *Introduction*

The concept of a neutrino radiation field is introduced (Griffiths & Newing, 1970) as being any neutrino field with energy momentum tensor

$$E_{\mu\nu} = A^2 l_\mu l_\nu$$

where l_μ is the neutrino flux vector. This definition is made by analogy to electromagnetic radiation fields. It has been shown (Audretsch, 1971) that all neutrino fields (with casual behaviour) become, asymptotically, neutrino radiation fields, a result which justifies physically the original definition.

Griffiths and Newing show that the Weyl equation for a neutrino field can be written in terms of a null tetrad as

$$(l_\mu m^\nu - l^\nu m_\mu)_{;\nu} = m^\alpha l_{\alpha;\mu} \tag{1.1}$$

where the null vector l^μ is the flux vector, satisfying

$$l^\mu{}_{;\mu} = 0 \tag{1.2}$$

The conditions for a neutrino radiation field are

$$m^\alpha l_{\alpha;\mu} = al_\mu + bm_\mu \tag{1.3}$$

and

$$\bar{m}^\alpha m_{\alpha;\mu} = -\tfrac{1}{2}iA^2 l_\mu + \bar{a}m_\mu - a\bar{m}_\mu \tag{1.4}$$

where b is real. The purpose of the present paper is to solve the Einstein field equations

$$R_{\mu\nu} = -A^2 l_\mu l_\nu \tag{1.5}$$

with the supplementary conditions (1.1) to (1.4). The method of spin coefficients, (Newman & Penrose, 1962) will be used throughout and readers unfamiliar with the definitions and notation are referred to the paper by Newman and Penrose. A coordinate system, adapted to the problem, is set up in Section 2 and the general solution to the resulting field equations is then obtained in the following sections. Three distinct classes of solution are obtained and these are characterised by the Petrov type of the Weyl tensor.

2. *Derivation of the Field Equations*

It follows from equation (1.3) that the neutrino flux vector is proportional to a gradient, $l_\mu = Au_{,\mu}$. It is convenient to introduce a new tetrad by the transformation

$$l_\mu \to A^{-1} l_\mu \quad \text{and} \quad n_\mu \to A n_\mu \tag{2.1}$$

so that the new vector l_μ (which is now no longer the neutrino flux vector) is *equal* to a gradient. This then leads to the following conditions on the spin coefficients

$$\tau = \bar{\alpha} + \beta, \qquad \rho = \bar{\rho}, \qquad \kappa = 0, \qquad \epsilon + \bar{\epsilon} = 0 \tag{2.2}$$

so that l_μ is tangent to a family of affinely parametrised null geodesics. Equations (1.1)–(1.4), after application of the transformation (2.1), yield

$$\tau = \bar{\alpha} - \beta, \qquad \epsilon = \bar{\epsilon} \tag{2.3}$$

together with the equations

$$2\tau - 2\beta = A_{,\nu} m^\nu / A \tag{2.4}$$

$$2\rho - 2\epsilon = A_{,\nu} m^\nu / A \tag{2.5}$$

and

$$\tfrac{1}{2} i \Lambda^2 A = \gamma - \bar{\gamma} \tag{2.6}$$

Coordinates $(x^1, x^2, x^3, x^4) \equiv (u, r, x^3, x^4)$ are now introduced, r being a preferred parameter for the null geodesics to which l_μ is tangent. It follows that

$$l^\mu = \delta_2{}^\mu \quad \text{and} \quad l_\mu = \delta_\mu{}^1$$

In order to preserve the orthonormality conditions on the null tetrad it is necessary to take

$$n^\mu = \delta_1{}^\mu + U\delta_2{}^\mu + X^i \delta_i{}^\mu$$

and

$$m^\mu = \omega\delta_2{}^\mu + \xi^i \delta_i{}^\mu$$

where indices i,j take the values 3 and 4. The metric can then be constructed using the completeness relations

$$g_{\mu\nu} = l_\mu n_\nu + l_\nu n_\mu - m_\mu \bar{m}_\nu - m_\nu \bar{m}_\mu \tag{2.7}$$

The metric and neutrino radiation equations (2.2)–(2.6) are invariant in form under the coordinate and tetrad transformations

$$r' = r + R(u, x^3, x^4) \tag{2.8}$$

$$x^{i'} = x^{i'}(u, x^3, x^4) \tag{2.9}$$

$$r' = r/\dot{\gamma}, \qquad u' = \gamma(u), \qquad l^{\mu'} = \dot{\gamma} l^\mu, \qquad n^{\mu'} = \dot{\gamma}^{-1} n^\mu \tag{2.10}$$

and

$$l^{\mu'} = l^\mu, \qquad n^{\mu'} = n^\mu + B\bar{m}^\mu + \bar{B}m^\mu + B\bar{B}l^\mu, \qquad m^{\mu'} = m^\mu + Bl^\mu \tag{2.11}$$

The intrinsic derivatives associated with the above null tetrad are

$$D = \partial/\partial r$$

$$\Delta = U\partial/\partial r + \partial/\partial u + X^i \partial/\partial x^i$$

and

$$\delta = \omega\, \partial/\partial r + \xi^i\, \partial/\partial x^i$$

Substituting the coordinates into the commution relations satisfied by these intrinsic derivatives yields, after simplification using (2.2) and (2.3),

$$D\xi^i = \rho\xi^i \tag{2.12}$$

$$D\omega = \rho\omega - \tau + \bar{\pi} \tag{2.13}$$

$$DX^i = (\tau + \bar{\pi})\,\bar{\xi}^i + (\bar{\tau} + \pi)\,\xi^i \tag{2.14}$$

$$DU = -(\gamma + \bar{\gamma}) + (\tau + \bar{\pi})\,\bar{\omega} + (\bar{\tau} + \pi)\,\omega \tag{2.15}$$

$$\delta U - \Delta\omega = -\bar{\nu} + \bar{\lambda}\bar{\omega} + (\mu - \gamma + \bar{\gamma})\,\omega \tag{2.16}$$

$$\delta X^i - \Delta\xi^i = (\mu - \gamma + \bar{\gamma})\,\xi^i + \bar{\lambda}\bar{\xi}^i \tag{2.17}$$

$$\delta\bar{\xi}^i - \bar{\delta}\xi^i = -\bar{\tau}\xi^i + \tau\bar{\xi}^i \tag{2.18}$$

$$\delta\bar{\omega} - \bar{\delta}\omega = -\bar{\tau}\omega + \tau\bar{\omega} + (\mu - \bar{\mu}) \tag{2.19}$$

Substituting (1.5), (2.2) and (2.3) into the Newman Penrose field equations gives

$$\psi_0 = \psi_1 = 0 \tag{2.20}$$

and

$$D\rho = \rho^2 \tag{2.21}$$

$$D\tau = (\tau + \bar{\pi})\,\rho \tag{2.22}$$

$$D\bar{\tau} = \rho\bar{\tau} + \rho\pi \tag{2.23}$$

$$D\gamma = (\tau + \bar{\pi})\,\bar{\tau} + \tau\pi + \psi_2 \tag{2.24}$$

$$D\lambda - \bar{\delta}\pi = \rho\lambda + \pi^2 + \bar{\tau}\pi \tag{2.25}$$

$$D\mu - \delta\pi = \rho\mu + \pi\bar{\pi} - \pi\tau + \psi_2 \tag{2.26}$$

$$D\nu - \Delta\pi = (\pi + \bar{\tau})\mu + (\bar{\pi} + \tau)\lambda + (\gamma - \bar{\gamma})\pi + \psi_3 \tag{2.27}$$

$$\Delta\lambda - \bar{\delta}\nu = -(\mu + \bar{\mu})\lambda - (3\gamma - \bar{\gamma})\lambda + (2\bar{\tau} + \pi)\nu - \psi_4 \tag{2.28}$$

$$\delta\rho = \rho\tau \tag{2.29}$$

$$\delta\bar{\tau} = \mu\rho + \tau\bar{\tau} - \psi_2 \tag{2.30}$$

$$\delta\lambda - \bar{\delta}\mu = (\mu - \bar{\mu})\pi + \mu\bar{\tau} + \bar{\lambda}\tau - \psi_3 \tag{2.31}$$

$$\delta\nu - \Delta\mu = \mu^2 + \lambda\bar{\lambda} + (\gamma + \bar{\gamma})\mu - \bar{\nu}\pi + \phi_{22} \tag{2.32}$$

$$\delta\gamma = \mu\tau + \bar{\tau}\bar{\lambda} \tag{2.33}$$

$$\delta\tau = \bar{\lambda}\rho \tag{2.34}$$

$$\Delta\rho - \bar{\delta}\tau = -\rho\bar{\mu} - 2\tau\bar{\tau} + (\gamma + \bar{\gamma})\rho - \psi_2 \tag{2.35}$$

$$\Delta\bar{\tau} - \bar{\delta}\gamma = \rho\nu - \tau\lambda - \bar{\tau}(\bar{\mu} - \bar{\gamma} + \gamma) - \psi_3 \tag{2.36}$$

Substituting (1.5), (2.2), (2.3) and (2.20) into the Bianchi identities yields

$$D\psi_2 = 3\rho\psi_2 \tag{2.37}$$

$$\delta\psi_2 = 3\tau\psi_2 \tag{2.38}$$

$$\bar{\delta}\psi_2 - D\psi_3 = -3\pi\psi_2 - 2\rho\psi_3 \tag{2.39}$$

$$\Delta\psi_2 - \delta\psi_3 = -3\mu\psi_2 - 2\tau\psi_3 - \rho\phi_{22} \tag{2.40}$$

$$\bar{\delta}\psi_3 - D\psi_4 = 3\lambda\psi_2 - 2(\bar{\tau} + 2\pi)\psi_3 - \rho\psi_4 \tag{2.41}$$

$$\Delta\psi_3 - \delta\psi_4 + \bar{\delta}\phi_{22} = 3\nu\psi_2 - 2(\gamma + 2\mu)\psi_3 - \tau\psi_4 - \bar{\tau}\phi_{22} \tag{2.42}$$

$$D\phi_{22} = 2\rho\phi_{22} \tag{2.43}$$

In the above

$$\phi_{22} = -\tfrac{1}{2}R_{\mu\nu}n^\mu n^\nu = +\tfrac{1}{2}A^2 \tag{2.44}$$

Under the null rotation (2.11), τ transforms as

$$\tau' = \tau + B\rho$$

It follows that the solutions of the equations (2.12)–(2.43) can be classified according as ρ is zero or non-zero. In the latter case the null rotation can be used to make τ zero. This case is considered in the next section.

3. *Neutrino Radiation Fields with Diverging Rays* (*i.e.* $\rho \neq 0$)

The null rotation (2.11) is used to make $\tau = 0$. It follows from (2.21) and (2.34) that $\pi = \lambda = 0$. Integrating (2.21) gives

$$\rho = -1/r \tag{3.1}$$

where a function of integration has been reduced to zero using the transformation (2.8). Equations (2.29), (2.19), (2.30), (2.38) and (2.31) yield

$$\omega = 0, \qquad \mu = \bar{\mu}, \qquad \psi_2 = \mu\rho, \qquad \delta\mu = 0, \qquad \psi_3 = 0 \tag{3.2}$$

Equations (2.12)–(2.15), (2.24)–(2.27), (2.39) and (2.40) can be solved to obtain the r-dependence of the unknowns. In what follows the superscript ^{0} is used to denote a function of $u\, x^3, x^4$ alone. Thus

$$\xi^i = \xi^{0i}/r \tag{3.3}$$

$$X^i = X^{0i} \tag{3.4}$$

$$\mu = \mu^0/r^2 \tag{3.5}$$

$$\gamma = \gamma^0 + \mu^0/2r^2 \tag{3.6}$$

$$U = U^0 - (\gamma^0 + \bar{\gamma}^0)\, r + \mu^0/r \tag{3.7}$$

$$\nu = \nu^0 \tag{3.8}$$

$$\psi_4 = \psi_4{}^0/r \tag{3.9}$$

$$\phi_{22} = \phi_{22}^0/r^2 \tag{3.10}$$

The transformation (2.9) is used to set $\xi^{03} = P$, $\xi^{04} = iP$. It then follows that

$$\xi^{0i}\, \partial/\partial x^i \equiv 2P\, \partial/\partial \bar{z}$$

where

$$z = x^3 + ix^4$$

The choice of the above canonical form for P does not exhaust the transformation (2.9), there still remains

$$z' = z'(z, u) \tag{3.11}$$

Equating to zero the different powers of r in equation (2.17) yields

$$U^0 = 0 \tag{3.12}$$

and

$$2PX^{0i}{}_{,\bar{z}} - (\gamma^0 + \bar{\gamma}^0)\, \xi^{0i} - \xi^{0i}{}_{,1} - X^{0j}\, \xi^{0i}{}_{,j} = (-\gamma^0 + \bar{\gamma}^0)\, \xi^{0i} \tag{3.13}$$

It follows from (3.13) that $X^0 = X^{03} + iX^{04}$ is independent of $\bar{z}$ and so can be reduced to zero by means of the transformation (3.11). Since X^{03}, X^{04} are real they must both vanish. Again the transformation (3.11) is not exhausted, there remains

$$z' = z'(z) \tag{3.14}$$

Equation (3.13) now becomes

$$2\bar{\gamma}^0 = -(\log P)_{,1} \tag{3.15}$$

The remaining equations amongst (2.12)–(2.43) are treated in the same way as equation (2.17) and the following information is obtained,

$$\nu^0 = \psi_4{}^0 = 0 \tag{3.16}$$

$$\gamma^0{}_{,z} = \gamma^0{}_{,\bar{z}} = P_{,z} = 0 \tag{3.17}$$

$$\mu^0{}_{,1} = -3(\gamma^0 + \bar{\gamma}^0)\mu^0 - \phi^0_{22} \tag{3.18}$$

$$\mu^0{}_{,z} = 0 \tag{3.19}$$

Equations (3.15) and (3.17) imply that

$$P = Z(\bar{z})\, U(u)$$

The transformation (3.14) can be used to make $Z(\bar{z}) = 1$ and the transformation (2.10) can be used to make $U(u) = \exp[i\chi(u)]$. Hence

$$P = \exp[i\chi(u)] \tag{3.20}$$

Substituting this into (3.15) and (3.18) yields

$$\bar{\gamma}^0 = -\tfrac{1}{2}i\dot{\chi} \tag{3.21}$$

$$\phi^0_{22} = -\dot{\mu}^0 \tag{3.22}$$

The equation (3.19) gives that the *real* function μ^0 is a function of u alone. The metric of the resulting space-times is obtained from (2.7) and is

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2\mu^0/r & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \tag{3.23}$$

Equations (2.4) and (2.5) yield

$$A = A^0(u)/r^2$$

so that the neutrino flux vector is

$$A^0(u)\, r^{-2}\, \delta_\mu{}^1 \tag{3.24}$$

Finally, comparing (3.22), (2.44) and (1.5) gives

$$R_{\mu\nu} = +2\dot{\mu}_0\, r^{-2}\, \delta_\mu{}^1\, \delta_\nu{}^1 \tag{3.25}$$

The only remaining equation, namely (2.6), yields $\dot{\chi} = \dot{\mu} A_0$. Notice that the neutrino flux vector is only defined up to an arbitrary multiplicative function of u and for a positive energy density μ_0 is a decreasing function of u. Since ψ_2 is the only non-zero tetrad component of the Weyl tensor it follows that the Weyl tensor is of type D. The corresponding empty space-time ($\mu^0 = \text{constant}$) is *not* flat.

4. *Neutrino Radiation Fields with Non-diverging Rays* (*i.e.* $\rho = 0$)

The null rotation (2.11) can be used to make $\pi = 0$; this fixes B up to $B = B(u, x^3, x^4)$. The r dependence of the various unknowns is easily obtained from the relevant equations, thus

$$\tau = \tau^0$$

$$\lambda = \lambda^0$$

$$\psi_2 = \psi_2{}^0$$
$$\phi_{22} = \phi_{22}^0$$
$$\xi^i = \xi^{0i}$$
$$\gamma = (\tau^0 \bar{\tau}^0 + \psi_2{}^0) r + \gamma^0$$
$$\mu = \psi_2{}^0 r + \mu^0$$
$$\omega = -\tau^0 r + \omega^0$$
$$X^i = (\tau^0 \bar{\xi}^{0i} + \bar{\tau}^0 \xi^{0i}) r + X^{0i}$$
$$U = U^0 + (\tau^0 \bar{\omega}^0 + \bar{\tau}^0 \omega^0 - \gamma^0 - \bar{\gamma}^0) r - \tfrac{1}{2}(4\tau^0 \bar{\tau}^0 + \psi_2{}^0 + \bar{\psi}_2{}^0) r^2$$

Equations (2.30), (2.34) and (2.35) become

$$\delta\bar{\tau}^0 = \tau^0 \bar{\tau}^0 - \psi_2{}^0 \tag{4.1}$$

$$\delta\tau^0 = 0 \tag{4.2}$$

$$-\bar{\delta}\tau^0 = -2\tau^0 \bar{\tau}^0 - \psi_2{}^0 \tag{4.3}$$

The coefficient of r in (2.33) is

$$\delta(\tau^0 \bar{\tau}^0 + \psi_2{}^0) = \psi_2{}^0 \tau^0$$

and this, together with (2.38) and (4.2) yields

$$\tau^0 \delta\bar{\tau}^0 = -2\psi_2{}^0 \tau^0 \tag{4.4}$$

Comparing (4.1), (4.3) and (4.4) yields

$$\tau^0 = \psi_2{}^0 = 0 \tag{4.5}$$

As in the previous section ξ^{03} and ξ^{04} can be put equal P and iP. The null rotation (2.11) can be used to make $\lambda^0 = 0$; this now fixes B up to $B = B(u, z)$. X^{0i} can then be transformed to zero, the remaining transformation (2.9) then being

$$z' = z'(z) \tag{4.6}$$

The following information is then readily obtained from equations (2.12)–(2.43).

$$\psi_3 = 2\bar{P}\mu^0{}_{,z} \tag{4.7}$$

$$\psi_4 = 2\bar{P}\psi^0_{3,z} r + 2\bar{P}\nu^0{}_{,z} + \bar{\omega}^0 \psi_3{}^0 \tag{4.8}$$

$$\phi_{22}^0 = -\mu^0{}_{,1} - \mu^0(\mu^0 + \gamma^0 + \bar{\gamma}^0) + \omega^0 \psi_3{}^0 + 2P\nu^0{}_{,\bar{z}} \tag{4.9}$$

$$\bar{\nu}^0 = \omega^0{}_{,1} + \omega^0(\mu^0 + 2\bar{\gamma}^0) - 2PU^0{}_{,\bar{z}} \tag{4.10}$$

$$\gamma^0{}_{,\bar{z}} = 0, \qquad P_{,z} = 0, \qquad \gamma^0{}_{,z} = \mu^0{}_{,z} \tag{4.11}$$

$$\mu_0 - \gamma^0 + \bar{\gamma}^0 = -(\log P)_{,1} \tag{4.12}$$

$$2P\bar{\omega}^0{}_{,\bar{z}} - 2\bar{P}\omega^0{}_{,z} = \mu^0 - \bar{\mu}^0 \tag{4.13}$$

Equations (2.4), (2.5) and (2.6) yield

$$A = A^0(u) \tag{4.14}$$

$$\tfrac{1}{2}i\Lambda^2 A^0 = \gamma^0 - \bar{\gamma}^0 = -iA^0 \phi^0_{22} \tag{4.15}$$

The interpretation of the equations (4.7)–(4.15) depends critically on whether $\psi_3 = 0$ or $\psi_3 \neq 0$. Consider first the case $\psi_3 = 0$. Then equation (4.7) gives

$$\mu^0 = \mu^0(\bar{z}, u)$$

and this function can be reduced to zero using the null rotation (2.11); this fixes B up to $B = B(u)$. γ^0 is now a function of u alone. From (4.12) $(\log P\bar{P})_{,1}$ = zero so that $P\bar{P}$ is a function of z, $\bar{z}$ alone. It follows, from this fact and (4.12), that

$$P = \bar{Z}(\bar{z}) \exp [i\chi(u)]$$

As in the last section $\bar{Z}(\bar{z})$ can be put equal to unity so that

$$P = \exp [i\chi(u)] \tag{4.16}$$

The transformation (2.10) can be used to make $\gamma^0 = -\bar{\gamma}^0$ and then equation (4.12) becomes

$$\gamma^0 = \tfrac{1}{2}i\dot{\chi} \tag{4.17}$$

So far the transformation (2.8) has not been used. Under this transformation

$$\omega^{0\prime} = \omega^0 + 2PR_{,\bar{z}} \tag{4.18}$$

The function R can be chosen to make $\omega^{0\prime}$ zero if and only if the integrability condition

$$\left(\frac{\omega^0}{P}\right)_{,z} = \left(\frac{\bar{\omega}^0}{\bar{P}}\right)_{,\bar{z}}$$

is satisfied. This is just equation (4.13) and so ω^0 can be put equal to zero. The only remaining equation is obtained by eliminating ϕ^0_{22} between (4.9) and (4.15). This gives an equation for U^0, namely

$$U^0{}_{,z\bar{z}} = m(u) = \tfrac{1}{4}A^0 \dot{\chi}$$

The general solution is

$$U^0 = m(u)\, z\bar{z} + F$$

where F is *any* function of u, z, $\bar{z}$ satisfying Laplace's equation $F_{,z\bar{z}} = 0$. The resulting metric is

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & mz\bar{z} + F & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \tag{4.19}$$

The neutrino flux vector is $A^0 \delta^1{}_\mu$ and

$$R_{\mu\nu} = -8m(u)\, \delta^1{}_\mu \delta^2{}_\mu$$

This metric is the plane fronted wave found previously (Audretsch & Graf, 1970). The neutrino flux vector is only defined up to an arbitrary multiplicative function of u. The Weyl tensor is of type N or O. The condition for conformal flatness (type O) is $F_{,zz} = 0$.

Now consider the case $\psi_3 \neq 0$. From (4.11) $\mu^0{}_{,z\bar{z}} = 0$ and so, using the null rotation (2.11), μ^0 can be put in the form

$$\mu^0 = \mu^0(z, u) \tag{4.20}$$

Equation (4.11) then gives

$$\gamma^0 = \mu^0(z, u) + \gamma^0(u) \tag{4.21}$$

It is now necessary to define a 'potential' $\bar{Q}(\bar{z}, u)$ by the equation

$$\frac{1}{P} = \frac{\partial \bar{Q}}{d\bar{z}} \tag{4.22}$$

It then follows that (4.13) can be written in the form

$$\left(\frac{\bar{\omega}^0}{\bar{P}} - \frac{\mu_0}{2\bar{P}}\bar{Q}\right)_{,\bar{z}} = \left(\frac{\omega^0}{P} - \frac{\bar{\mu}^0}{2P}Q\right)_{,z}$$

and this equation is the integrability condition for the function R in equation (4.18) to be chosen so that

$$\omega^0 = \tfrac{1}{2}\bar{\mu}^0 Q \tag{4.23}$$

Equation (4.12) can now be used to obtain $\bar{\mu}^0(\bar{z}, u)$ in terms of $\bar{Q}(\bar{z}, u)$ and $\gamma^0(u)$

$$\bar{\mu}^0(\bar{z}, u) = \gamma^0 - \bar{\gamma}^0 + (\log \bar{Q}_{,\bar{z}})_{,1} \tag{4.24}$$

The only remaining equation is obtained by eliminating ϕ_{22}^0 between (4.9) and (4.15). This equation is

$$4P\bar{P}U^0{}_{,z\bar{z}} = \mu^0\bar{\mu}^0 + \bar{P}Q\bar{\mu}^0\mu^0{}_{,z} + P\bar{Q}\mu^0\bar{\mu}^0{}_{,\bar{z}} - iA^0(\mu^0 - \bar{\mu}^0 + \gamma^0 - \bar{\gamma}^0)$$

This defines U^0 in terms of $\bar{Q}(\bar{z}, u)$, $\gamma^0(u)$ and $A^0(u)$ up to an arbitrary solution of Laplace's equation $U^0{}_{,z\bar{z}} = 0$. It follows that the metric of the space-times under discussion here are generated by $\bar{Q}$, γ^0 and A^0. The neutrino flux vector is $A^0\delta_\mu{}^1$ and since A^0 appears explicitly in the metric the arbitrariness in the flux vector which occurred in the previous two cases no longer occurs here. This result is in agreement with a theorem due to Griffiths & Newing (1971). The Weyl tensor for this third class of space-times is of type III.

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